8.1 Linearity of expectation:

build up to CLT and LLN

Now that we have covered joint pdf and joint pmfs, we will cover 3 important concepts:

1) Expectation of sums (Linearity of Expectation)

2) Variance of sums

3) Covariance.

$$y = mx = f(x)$$

$$f(ax+by) = m(ax+by) = a(mx) + b(my)$$

$$= af(x) + bf(y)$$

linear functions of this form satisfy the linearity property.

Linearity: $f(ax+by) = af(x) + bf(y)$

Ex: let X and Y represent the values of two dice rolls. I want to find the expected value

$$E[aX + bY^2]$$

Linearity tells you

$$= aE[X] + bE[Y^2]$$

We will prove linearity in this example.

$$g(x,y) = ax + by^2$$

$$E[g(X,Y)] = \sum_{i,j} (ai + bj^2)\, p_{X,Y}(i,j)$$

$$= \underbrace{\sum_{i=1}^{6}}_{\text{range } X} \underbrace{\sum_{j=1}^{6}}_{\text{range } Y} (ai + bj^2) \cdot \underbrace{\frac{1}{6} \cdot \frac{1}{6}}_{\text{independence.}}$$

$$E[g(X,Y)] = \sum_{\substack{i \in \text{range}(X) \\ j \in \text{range}(Y)}} g(i,j)\, p_{X,Y}(i,j)$$

split up the summands.

$$= \overbrace{\underbrace{\sum_{j=1}^{6}\sum_{i=1}^{6} a\,i \cdot \frac{1}{6}\cdot\frac{1}{6}}_{P}} + \underbrace{\sum_{j=1}^{6}\sum_{i=1}^{6} b\,j^2 \frac{1}{6}\cdot\frac{1}{6}}_{Q}$$

only depends on i

$$P = \sum_{i=1}^{6} a\,i \cdot \frac{1}{6} \underbrace{\left(\sum_{j=1}^{6} \frac{1}{6}\right)}_{1} = \sum_{i=1}^{6} a\,i \cdot \frac{1}{6}$$

$$= a \underbrace{\sum_{i=1}^{6} i \frac{1}{6}} = a E[X].$$

definition of expectation of X.

Similarly $Q = bE[Y^2]$

started here $\to$ $\mathbb{E}[aX + bY^2] = \underbrace{a\,\mathbb{E}[X] + b\,\mathbb{E}[Y^2]}$

finished here.

Nowhere in this calculation have we used that X and Y came from independent die rolls.

$\hookrightarrow$ we did, but the computation is the same in general.

Let us state this as a theorem:

<u>Linearity</u>: Let $g_1, \dots, g_n$ be single variable functions $X_1, \dots, X_n$ be rvs. Then

Expectation of a sum = Sum of expectations.

$$E[g(X_1) + \cdots + g(X_n)] = E[g(X_1)] + \cdots + E[g(X_n)]$$

In the above example we had (X, Y)

$$g_1(x) = ax \qquad g_2(y) = by^2$$

Ex: Binomial(n,p) $\sim S_n$ = "# of successes in n trials"

$S_3 = 1 + 0 + 1$ = "total of 2 success"

<u>Binomial is a sum of indicators</u>

$\{0,1\} \ni X_1$

$$X_i = \begin{cases} 1 & \text{if heads on } i^{th} \text{ coin.} \\ 0 \end{cases}$$ (i^{th} trial is a success)

$$X_1 + \cdots X_n \stackrel{?}{=} S_n \text{ (binomial)}$$

$$E[S_n] = E[X_1 + \cdots X_n] \xrightarrow{\text{linearity}} = E[X_1] + \cdots E[X_n] = p + \cdots + p$$

$$= np = E[S_n] \; !$$

A Binomial is a sum of (independent) Bernoullis

We have mentioned this earlier, but it is worth repeating.

$$X_1 = \begin{cases} 1 & \text{with prob } p \\ 0 \end{cases} \qquad E[X_1] = P(X_1 = 1) = p$$

$$Var(S_n) = np(1-p) \stackrel{?}{=} Var(X_1) + \cdots + Var(X_n)$$

$$Var(X_1) = p(1-p)$$ (Variance of a Bernoulli)

Ex 8.6

There is a party every month in an office with 15 coworkers. There is ONE party at the end of the month if ANY coworker has a birthday in that month. Find the average # of birthdays parties in a year.

Stingy management; they don't want to pay for 15 parties. So they want to combine several birthdays into one by having a single party at the end of the month.

$$E[\#\text{ of birthday parties}]$$

J_1 is 1 if there is a party in month 1.

$$= E[J_1 + J_2 + \cdots + J_n]$$

We would like to set this average up as a sum of random variables so we can use linearity of expectation.

How do you set this up as a sum?

Y = total # of parties in a year.

$Y = J_1 + J_2 + \cdots + J_{12}$

J_1 = # of parties in month 1.

Each J_i takes the values 0 or 1.

Is J_1 independent of J_2?

10 Yes | POLL | No. ? → 2

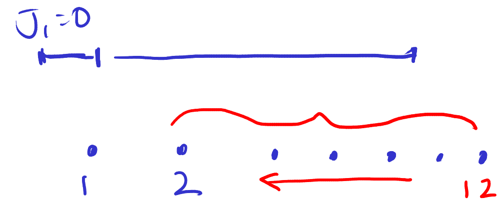
Let's compute.

$P(J_1 = 0) = \left(\frac{11}{12}\right)^{15}$ → all employees do not have a birthday in month 1.

All 15 individuals do not have a birthday in month 1.

$P(J_1 = a, J_2 = b) \stackrel{?}{=} P(J_1 = a)P(J_2 = b)$ for all a, b

$$P(J_2=0,\ J_1=0) = P(J_2=0 \mid J_1=0)\,P(J_1=0)$$



$$P(J_2=0 \mid \overbrace{J_1=0}) = \left(\frac{10}{11}\right)^{15}$$

(One has to think about this)

So $$P(J_2=0,\ J_1=0) = \left(\frac{10}{11}\right)^{15}\left(\frac{11}{12}\right)^{15}$$

$$\neq P(J_2=0)\,P(J_1=0)$$

$$=$$

BUT

$$\mathbb{E}[Y] = \sum_{i=1}^{12}\mathbb{E}[J_i] = 12\cdot\mathbb{E}[J_1]$$

$$\mathbb{E}[J_1] = 1\cdot P(J_1=1) + 0\cdot P(J_1=0)$$

$$= 1 - \left(\frac{11}{12}\right)^{15}.$$

(15) = $\Omega = \{ (a_1, \dots, a_{15}) : a_i \in \{1, 2, \dots, 12\} \}$

Birthday month of employee i

$= |[12]| \times |[12]| \cdots \times |[12]| = 12^{15}$

$P(J_1 = 0) = |\{ (a_1, \dots, a_{15}) : a_i \in \{2, \dots, 12\}\}| / |\Omega|$

no one has a birthday in month 1.

$= \frac{11^{15}}{12^{15}} = \left(\frac{11}{12}\right)^{15}$

$P(J_2 = 0, J_1 = 0) = \frac{| \{ (a_1, \dots, a_{15}) : a_i \in \{3, \dots, 12\}\} |}{(12)^{15}}$

$= \frac{10^{15}}{12^{15}}$

$\neq \underbrace{P(J_2 = 0)}_{(11/12)^{15}} \underbrace{P(J_1 = 0)}_{(11/12)^{15}}$

2.

POLL

Ex: Manu must pass both a written test and a road test to get a drivers licence. Each time he takes the written test, he passes with a probability $\frac{4}{10}$ independently of other tests. Each time he takes the road test, he passes with probability $\frac{7}{10}$, again independently of other tests. What is the total expected # of tests he must take before earning his licence?

Hint: let X_1 be the number of written tests he takes before passing, and let X_2 be the # of road tests.

Ex: (Another version of the indicator method)

The sequence of coin flips HHTHHHTTTHHHT
contains two runs of heads of length 3
one run of heads of length 2
no runs of heads of length 1

If a fair coin is flipped 100 times, what is the expected # of runs of heads of length 3.

$$\Omega = \{H,T\}^{100} = \{(\omega_1, \dots \omega_{100}) : \omega_i \in \{H,T\}\}$$

let $I_k = \{$A run of heads of length 3 begins at toss $k\}$

$$= \{\omega : \omega_k, \omega_{k+1}, \omega_{k+2} = H, \underbrace{\omega_{k-1} = T}_{\text{k cannot work if } k=1}, \underbrace{\omega_{k+4} = T}_{\text{cannot work if } k=97}\}$$

$$I_1 + I_{97} + \sum_{k=1}^{96} I_k = \text{\# of runs of length 3.}$$

$$E[I_k] =$$

$$E[I_{97}] =$$

$$E[I_1] =$$